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Directed Compact Lattice Animals: Exact Results

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The partially directed compact lattice animal model on the square lattice is solved exactly for the cluster number and average cluster radius along the directed axis in terms of the appropriate generating functions. For the critical exponents we find $\theta = 0$ and $\nu_{\|} = 1$. Caliper size distribution along the directed axis is also calculated analytically. It is used to confirm $\nu_{\|} = 1$ and to study some finite-size scaling properties for this model. For the perpendicular cluster radius distribution, a combination of analytic arguments and computer results leads to a conjecture on the exact form of the appropriate generating function and to the result $\nu_{\perp} = \frac{1}{2}$. Some calculations are reported for the triangular lattice and for hypercubic lattices in $d > 2$.

KEY WORDS: Lattice animals; cluster radius exponents; directed clusters; finite-size effects.

1. INTRODUCTION

Recently, we reported[1] exact derivation of the number generating function for the partially directed compact lattice animal model on the square lattice. In the present work, we summarize new comprehensive studies of this model, including new analytic results for several cluster radius generating functions, and examine finite-size scaling properties and also some extensions to $d > 2$. A characteristic large-N asymptotic growth law for the number of distinct connected N-site clusters (lattice animals) reads

$$c_N \sim N^{-\theta} \lambda^N \tag{1.1}$$

while the cluster radii increase according to

$$R_{\|}(N) \sim N^{\nu_{\|}} \qquad \text{and} \qquad R_{\perp}(N) \sim N^{\nu_{\perp}} \tag{1.2}$$

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These quantities will be defined precisely in the following sections. Our results for directed compact animals are

$$\theta = 0, \qquad \nu_{\parallel} = 1, \qquad \text{and} \qquad \nu_{\perp} = \tfrac{1}{2} \tag{1.3}$$

The observation that compactness and directional constraints, when combined, may lead to solvable systems was reported by Derrida and Nadal.[2] Their model is, however, too restrictive in that the asymptotic behavior differs from the generic laws (1.1)–(1.2). Bhat *et al.*[3] considered the fully directed compact lattice animals on the square lattice. This model has not been solved analytically, although high-precision numerical studies were reported.[1,3] For a *partially directed* version, generating function techniques can be used to obtain exact results.[1]

In Section 2, we define the model and introduce a generating function formalism that encompasses the results of Ref. 1, but also permits calculation of a parallel cluster size measure $R_{\parallel}$, thus leading to $\theta = 0$ and $\nu_{\parallel} = 1$. Derivation of the parallel caliper size distribution is reported in Section 3. It is used to confirm $\nu_{\parallel} = 1$ and in Section 4 is employed to investigate the form of the finite-size scaling for this model. Section 5 is devoted to studying the perpendicular cluster size measures. Analytic arguments and numerical results are combined to conjecture the exact form of the appropriate generating function, leading to $\nu_{\perp} = \frac{1}{2}$. In Sections 6 and 7, we consider the triangular and hypercubic lattices, respectively. Exact results are reported for the number generating functions; universality and dependence on the dimensionality d are discussed.

The exponent values and the form of the finite-size scaling for compact animals are similar to those for directed walks. Indeed, we find that the structure of these animals is walklike in several aspects; Section 5 gives details and discussion.

2. GENERATING FUNCTION FORMALISM

Our model is defined on the square lattice with spacing 1. The origin $(X, Y) = (0, 0)$ is a site in every cluster. The remaining $N - 1$ sites must be reachable from the origin by a partially directed walk of nearest neighbor steps in the $+X$ and $\pm Y$ directions, between cluster sites. We also add the condition that the point $(0, -1)$ is not in a cluster, to avoid counting animals that differ only by overall translations. Finally, *compactness* is imposed at each "time" level, i.e., for fixed values of the directed coordinate X, by requiring that all cluster sites with the same X form a sequence of nearest neighbors. (There is no restriction in the case of a single site at a given X.)

Let $c_N(k)$ denote the number of distinct N-site animals having exactly k "root" sites at $X=0$, with $Y=0, 1,..., k-1$. Thus, $c_N(k)=0$ for $N<k$ and $c_k(k)=1$. For each N-site, k-root animal, let x_n denote the X coordinates of the sites: $n=1, 2,..., N$. With the index a used to label all the k-root animals, we define the generating function

$$F_k(z, u)=\sum_a z^{N(a)-k} u^{\sum_{n=1}^{N(a)} x_n(a)} \tag{2.1}$$

where $N(a)$ is the number of sites in the ath animal and $x_n(a)$ are the appropriate X coordinates. Since

$$F_k(z, 1)=\sum_{N=k}^{\infty} c_N(k)\, z^{N-k} \tag{2.2}$$

the calculation of $F_k(z, 1)$, accomplished in Ref. 1, leads to results on the behavior of $c_N(k)$.

One way of defining the cluster size measure along the X axis is by averaging the center-of-mass X coordinate,

$$\frac{1}{N}\sum_{n=1}^{N} x_n \tag{2.3}$$

over N-site animals. With the index b used to label all the N-site clusters, we have

$$R_{\|}(N)=c_N^{-1}\sum_{b=1}^{c_N}\left[N^{-1}\sum_{n=1}^{N} x_n(b)\right] \tag{2.4}$$

so that

$$Nc_N R_{\|}(N)=\sum_{b=1}^{c_N}\sum_{n=1}^{N} x_n(b) \sim N^{\nu_{\|}-\theta+1}\lambda^N \tag{2.5}$$

We can also define $R_{\|}(N, k)$ for k-root clusters only. The relevant generating function takes the form

$$f_k(z) \equiv \left[\frac{\partial F_k(z, u)}{\partial u}\right]_{u=1}=\sum_{N=k}^{\infty} Nc_N R_{\|}(N, k)\, z^{N-k} \tag{2.6}$$

We will now proceed to calculate the left sides of (2.2) and (2.6). Generating functions for c_N and $Nc_N R_{\|}(N)$ without a prescribed root size will also be obtained. Let us emphasize that the first-moment cluster size measures, as in (2.4), vanish identically for symmetric axes. Thus, analogous definition is not possible for $R_{\perp}(N)$: see Section 5 for further discussion.

For each $(N \geqslant k+1)$-site animal with k root at $X=0$, the $N-k$ sites not in the root form a smaller animal with a root of m sites at $X=1$. This m root is not pinned with its lowest Y site at $Y=0$, but can occur in $k+m-1$ different locations, as implied by the connectivity rules. Let the index a label the *originally defined* m-root animals, i.e., with the root at $X=0$ and its lowest site at $Y=0$. Then we can write the following relation for F_k defined in (2.1):

$$F_k(z, u) = 1 + \sum_{m=1}^{\infty} (k+m-1) \sum_{a} z^{N(a)} u^{\sum_{n=1}^{N(a)} [x_n(a)+1]} \tag{2.7}$$

where the first term corresponds to the k root itself. The multiplicity factor $(k+m-1)$ was explained above, and the x_n+1 are used to account for the shift of the m root to $X=1$. By inspection of the appropriate defining relations, one can check that (2.7) is summarized by

$$F_k(z, u) = 1 + \sum_{m=1}^{\infty} (k+m-1)(zu)^m F_m(zu, u) \tag{2.8}$$

The $u \equiv 1$ relation,

$$F_k(z, 1) = 1 + \sum_{m=1}^{\infty} (k+m-1)\, z^m F_m(z, 1) \tag{2.9}$$

was considered in Ref. 1. One is ultimately interested in the quantity

$$A(z) = \sum_{N=1}^{\infty} c_N z^N = \sum_{n=1}^{\infty} z^n F_n(z, 1) \tag{2.10}$$

which generates the total number of N-site animals c_N. Rearrangement of (2.9) yields

$$F_k(z, 1) = kA(z) + B(z) \tag{2.11}$$

where all the k dependence is displayed. The functions $A(z)$ and $B(z)$ can be calculated[1] by substituting (2.11) in the $k=1$ and $k=2$ relations (2.9). Then the $k>2$ relations are satisfied automatically[1] by (2.11). The results are

$$A(z) = z(1-z)^3/(1-5z+7z^2-4z^3) \tag{2.12}$$

$$B(z) = (1-z)^2(1-3z+z^2)/(1-5z+7z^2-4z^3) \tag{2.13}$$

The singularity of $A(z)$ [and $B(z)$] nearest to the origin is a simple pole at $z_c = \lambda^{-1} < 1$, where

$$\lambda = 3.20556943040\ldots \tag{2.14}$$

[for the exact expression for λ see Ref. 1 or Eq. (7.5)]. The asymptotic form (1.1) for c_N corresponds to the $(1-\lambda z)^{\theta-1}$ singularity in the generating function $A(z)$; thus, $\theta = 0$ for this model. Detailed calculations of the behavior of c_N and fixed-root $c_N(k)$ as $N \to \infty$ were presented in Ref. 1.

We now proceed to calculate the derivative appearing on the left side of (2.6). Relation (2.8) can be replaced by

$$F_k\left(\frac{z}{u}, u\right) = 1 + \sum_{m=1}^{\infty} (k+m-1)\, z^m F_m(z, u) \tag{2.15}$$

Differentiation with respect to u and the substitution $u = 1$ lead to

$$f_k(z) = z\,\frac{dF_k(z, 1)}{dz} + \sum_{m=1}^{\infty} (k+m-1)\, z^m f_m(z) \tag{2.16}$$

where $f_k(z)$ were defined in (2.6). Since both terms on the right are linear in k, we conclude that

$$f_k(z) = kC(z) + D(z) \tag{2.17}$$

The functions $C(z)$ and $D(z)$ can be calculated by substitution in the first two relations (2.16). A tedious algebraic calculation yields

$$C(z) = \frac{z(1-z)^4(1-2z)(1-4z+10z^2-8z^3+8z^4-2z^5)}{(1-5z+7z^2-4z^3)^3} \tag{2.18}$$

$$D(z) = \frac{2z^2(1-z)^3(1-5z+10z^2-12z^3+15z^4-7z^5+2z^6)}{(1-5z+7z^2-4z^3)^3} \tag{2.19}$$

The generating function for quantities $Nc_N R_{||}(N)$ defined by (2.5) can be expressed in terms of $f_k(z)$ as follows:

$$E(z) \equiv \sum_{N=1}^{\infty} [Nc_N R_{||}(N)]\, z^N = \sum_{k=1}^{\infty} z^k f_k(z) \tag{2.20}$$

which, by (2.17), reduces to

$$\begin{aligned} E(z) &= z(1-z)^{-2}[C(z) + (1-z)\, D(z)] \\ &= \frac{z^2(1-z)^2(1-4z+8z^2-8z^3+12z^5-10z^6+4z^7)}{(1-5z+7z^2-4z^3)^3} \end{aligned} \tag{2.21}$$

The asymptotic behavior in (2.5) corresponds to the $(1-\lambda z)^{-(\nu_{\|}+2-\theta)}$ singularity in $E(z)$. Since $E(z)$ has a cubic pole at z_c, we conclude that

$$\nu_{\|}=1 \tag{2.22}$$

for the compact directed animal model.

3. CALIPER SIZE DISTRIBUTION

In this section we consider the caliper size distribution along the directed axis X. Let $c_{N,L}(k)$ denote the number of distinct N-site, k-root animals with exactly L columns, i.e., with X ranging from 0 for the "root" column to $L-1$ for the last column. The following properties are straightforward:

$$c_{N,L}(k)=\begin{cases} 0, & N<k+(L-1) \\ k, & N=k+(L-1) \\ \delta_{Nk}, & L=1 \end{cases} \tag{3.1}$$

Recursion relations for $c_{N,L}(k)$ can be derived by considering the connectivity rules formulated in Section 2. We have

$$c_{N,L}(k)=\sum_{m=1}^{N-k-(L-2)} (k+m-1)\, c_{N-k,L-1}(m) \tag{3.2}$$

where the upper limit on m is obtained from the "conservation of sites" condition

$$m_{\max}+(L-2)=N-k \tag{3.3}$$

Note that (3.2) is consistent with (3.1).

It is useful to introduce the double-generating function

$$G_k(z,v)=\sum_{L=1}^{\infty}\ \sum_{N=k+(L-1)}^{\infty} c_{N,L}(k)\, z^{N-k} v^L \tag{3.4}$$

By using (3.1), we have

$$G_k(z,v)-v=\sum_{L=2}^{\infty}\ \sum_{N=k+(L-1)}^{\infty} c_{N,L}(k)\, z^{N-k} v^L \tag{3.5}$$

Substitution of (3.2) in the right side of this relation and rearrangement of the resulting triple sum yields

$$G_k(z,v)=v\left[1+\sum_{m=1}^{\infty}(k+m-1)\, z^m G_m(z,v)\right] \tag{3.6}$$

This relation is similar to (2.9): the k dependence is linear,

$$G_k(z, v) = v[kA(z, v) + B(z, v)] \tag{3.7}$$

The extended functions $A(z, v)$ and $B(z, v)$ reduce to A and B of Section 2 at $v = 1$ [i.e., $A(z, 1) = A(z)$, etc.]. These functions can be calculated by substituting (3.7) in the $k = 1, 2$ relations (3.6). A long calculation leads to

$$A(z, v) = \frac{vz(1-z)^3}{[1-(4+v)z+(6+v)z^2-(4-v+v^2)z^3+(1-v)z^4]} \tag{3.8}$$

$$B(z, v) = \frac{(1-z)^2[(1-z)^2 - vz]}{[1-(4+v)z+(6+v)z^2-(4-v+v^2)z^3+(1-v)z^4]} \tag{3.9}$$

One possible definition of the parallel cluster size measure $r_{\|}(N)$ is given by the first moment of the spanning size L,

$$c_N r_{\|}(N) = \sum_{L=1}^{N} L c_{N,L} \sim N^{\nu_{\|} - \theta} \lambda^N \tag{3.10}$$

where

$$c_{N,L} \equiv \sum_{k=1}^{N-(L-1)} c_{N,L}(k) \tag{3.11}$$

The appropriate generating function is defined as

$$H(z) \equiv \sum_{N=1}^{\infty} [c_N r_{\|}(N)] z^N = \sum_{N=1}^{\infty} \sum_{L=1}^{N} \sum_{k=1}^{N-(L-1)} L c_{N,L}(k) z^N \tag{3.12}$$

Rearranging the sums, one can show that

$$H(z) = \left[\frac{\partial}{\partial v} \sum_{k=1}^{\infty} z^k G_k(z, v)\right]_{v=1} = \left[\frac{\partial}{\partial v} A(z, v)\right]_{v=1} \tag{3.13}$$

where the identification of the k sum with $A(z, v)$, analogous to (2.10), follows by examination of the k term in (3.6). Thus, we get

$$H(z) = \frac{z(1-z)^3(1-4z+6z^2-3z^3+z^4)}{(1-5z+7z^2-4z^3)^2} \tag{3.14}$$

confirming $\nu_{\|} = 1$. Indeed, the asymptotic form in (3.10) corresponds to the $(1-\lambda z)^{-(\nu_{\|} - \theta + 1)}$ singularity in $H(z)$.

4. FINITE-SIZE SCALING PROPERTIES

In the preceding section we found that the quantity $A(z, v)$ given by (3.8) can be represented as

$$A(z, v) \equiv \sum_{k=1}^{\infty} z^k G_k(z, v) = \sum_{L=1}^{\infty} v^L \sum_{N=L}^{\infty} c_{N,L} z^N \tag{4.1}$$

where $c_{N,L}$ are defined by (3.11). If we regard the z^N factor as fugacity weight in a grand-canonical-type ensemble, then $A(z, v)$ "generates" the fixed-L partition functions

$$z_L(z) \equiv \sum_{N=L}^{\infty} c_{N,L} z^N \tag{4.2}$$

It can be used to calculate thermodynamic quantities. Furthermore, one possible definition of the parallel correlaltion length is

$$\xi_{\|}(z) = \frac{H(z)}{A(z, 1)} = \left[\frac{\partial \ln A(z, v)}{\partial v}\right]_{v=1} \approx (z_c - z)^{-1} \tag{4.3}$$

where the asymptotic divergence with exponent $\nu_{\|} = 1$ follows from the explicit results of Section 3; see (2.12), (3.12), and (3.14).

Consider now a system of *finite* extent M along the x axis. The cluster sites can only have X coordinates 0, 1,..., $M-1$. The appropriate generating function for this finite-size problem is analogous to $A(z, v)$ in (4.1), but with the L values restricted to $L \leqslant M$,

$$A_M(z, v) \equiv \sum_{L=1}^{M} v^L Z_L(z) \tag{4.4}$$

The form of the large-L asymptotic behavior of thermodynamic and correlation quantities for z near $z_c \equiv 1/\lambda$ is described by the finite-size scaling,[4] the formulation of which for anisotropic systems was reviewed, e.g., in Ref. 5. Specifically, for the animal number generating function considered in Section 1, we expect

$$\frac{A_M(z, 1)}{A_\infty(z, 1)} \approx P(\tau) \tag{4.5}$$

where the scaling combination τ is defined by

$$\tau \equiv M/\xi_{\|}(z) \sim (z_c - z) M \tag{4.6}$$

Similarly, if we define a finite-system, correlation length-like quantity

$$\xi_M(z) \equiv \left[\frac{\partial \ln A_M(z, v)}{\partial v}\right]_{v=1} \tag{4.7}$$

then

$$\xi_M(z)/\xi_{||}(z) \approx Q(\tau) \tag{4.8}$$

The scaling functions $P(\tau)$ and $Q(\tau)$ are universal.[6] We will focus on the ratio (4.5), establish the scaling relation, and calculate the scaling function $P(\tau)$. It will be apparent, however, that the calculation can be extended to (4.8) and other quantities defined in the grand canonical ensemble.

The $M = \infty$ generating function $A(z, v)$ given by (3.8) can be represented in the form

$$A(z, v) = \frac{(1-z)^3}{z^2} [v_-(z) - v_+(z)]^{-1} \left[\frac{1}{1 - v/v_-(z)} - \frac{1}{1 - v/v_+(z)}\right] \tag{4.9}$$

where

$$v_\pm(z) = -\frac{(1-z)^2}{2z^2} [1 + z \pm (1 + 6z + z^2)^{1/2}] \tag{4.10}$$

are the roots of the denominator of (3.8). It follows that the fixed-L partition functions defined in (4.4) are given by

$$Z_L(z) = \frac{1-z}{(1+6z+z^2)^{1/2}} [v^{-L}_-(z) - v_+^{-L}(z)] \tag{4.11}$$

This result can be used to calculate *exactly* various finite-M quantities. However, for the scaling analysis it is convenient to work with the large-L expressions only. Therefore, we consider the difference [compare (4.5)]

$$A_\infty(z, 1) - A_M(z, 1) = \sum_{L=M+1}^{\infty} Z_L(z) \tag{4.12}$$

The functions $v_\pm(z)$ given by (4.10) have the following property for $0 < z < z_c$:

$$-v_+(z) > v_-(z) > 1 \tag{4.13}$$

At z_c, we have

$$-v_+(z_c) > v_-(z_c) = 1 \tag{4.14}$$

For large M and L ($>M$), the contribution to (4.12) due to the v_+ term in (4.11) constitutes an exponentially small (oscillating) correction, which does not contribute to the leading, scaling behavior. The v_- contribution, however, is *divergent* as $z \to z_c^-$. For the scaling description, we can use

$$A_\infty(z, 1) - A_M(z, 1) \approx \frac{1-z}{(1-6z+z^2)^{1/2}} \frac{v_-^{-M}(z)}{v_-(z)-1} \tag{4.15}$$

In the limit $z \to z_c^-$ the denominator of (4.15) vanishes, while the v_-^{-M} term can be represented as

$$v_-^{-M}(z) = \exp[-M \ln v_-(z)] \approx e^{-k\tau} \tag{4.16}$$

since $\ln v_-(z)$ has a simple zero at z_c. Near z_c, $\ln v_-$ can be replaced by $v_- - 1$ and the constant k evaluated as

$$\begin{aligned} k &= \lim_{z \to z_c} \{[v_-(z) - 1]\, \xi_{\|}(z)\} \\ &= \left[\frac{4(1-4z+6z^2-3z^3+z^4)}{z(1+z+\zeta)(2-5z+z^2+\zeta z)}\right]_{z_c} = 1 \end{aligned} \tag{4.17}$$

where

$$\zeta = \zeta(z) \equiv (1+6z+z^2)^{1/2} \tag{4.18}$$

Note that all the factors in the rational expression are finite at z_c, and that the last two steps in (4.17) each requires substantial algebraic calculations. Inspection of the relations (4.15), (4.16), and (2.12) leads to the scaling form (4.6) with

$$P(\tau) = 1 - Ke^{-\tau} \tag{4.19}$$

The constant K is given by

$$\begin{aligned} K &= \lim_{z \to z_c} \frac{1-z}{\zeta(z)[v_-(z)-1]\, A(z, 1)} \\ &= \left[\frac{(1+z+\zeta)(2-5z+z^2+\zeta z)}{4\zeta(1-z)^2}\right]_{z_c} = 1 \end{aligned} \tag{4.20}$$

The comments following (4.18) apply here as well. In summary, the universal finite-size scaling function $P(\tau)$ takes the form

$$P(\tau) = 1 - e^{-\tau} \tag{4.21}$$

5. PERPENDICULAR CORRELATION EXPONENT

The definition of cluster radius measures along the Y axis requires the use of the second or higher even-power moments of the cluster size distribution, because of the $\pm Y$ symmetry. One possible definition, reminiscent of the radius-of-gyration forms used for isotropic animals, is introduced as follows. Let the index a label all the N-site animals and $y_n(a)$ denote the Y coordinates of the sites in the ath cluster, $n = 1, 2,..., N$. We define $R_\perp(N)$ via

$$N^2 c_N R_\perp^2(N) = \sum_{a=1}^{c_N} \sum_{n=1}^{N} N^2 \left[y_n(a) - \frac{1}{N} \sum_{m=1}^{N} y_m(a) \right]^2 \tag{5.1}$$

Thus, $R_\perp(N)$ is the root-mean-squared deviation from the center-of-mass Y coordinate. The factor N^2 is introduced to make the left side of (5.1) an integer number.

We were not able to find a recursion relation for the appropriate generating function

$$W(z) = \sum_{n=1}^{\infty} [N^2 c_N R_\perp^2(N)]\, z^N \tag{5.2}$$

However, all the generating functions encountered thus far have had a similar pattern of singularity structure: an integral power of the polynomial $(1 - 5z + 7z^2 - 4z^3)$ in the denominator [see (2.12)]. Thus, we conjecture that $W(z)$ takes the form

$$W(z) = \frac{w(z)}{\omega(z)(1 - 5z + 7z^2 - 4z^3)^p} \tag{5.3}$$

where $w(z)$ and $\omega(z)$ are polynomials.

We generated the numbers $N^2 c_N R_\perp^2(N)$ numerically for $N = 1, 2,..., 18$ by direct enumeration of all possible clusters. An appropriately modified version of a standard computer algorithm[7] was used. The results are summarized in Table I. From the first 18 terms in the power series expansion of $W(z)$, analogous series for $(1 - 5z + 7z^2 - 4z^3)^p\, W(z)$ were generated for $p = 1, 2, 3, 4$. The $p = 4$ series has the last few terms constant, suggesting $\omega(z) = 1 - z$ [see (5.3)]. Finally, the series for $(1 - z)(1 - 5z + 7z^2 - 4z^3)^4\, W(z)$ has zero $N = 14, 15,..., 18$ entries. These computer results are summarized by

$$\begin{aligned} W(z) = z^2[1 &- 7z + 30z^2 - 108z^3 + 283z^4 - 581z^5 + 1100z^6 \\ &- 1800z^7 + 2093z^8 - 1537z^9 + 632z^{10} - 120z^{11} \\ &+ o(z^{16})][(1 - z)(1 - 5z + 7z^2 - 4z^3)^4]^{-1} \end{aligned} \tag{5.4}$$

Table I. Enumeration Data for $N^2 c_N R_\perp^2(N)$, $N=1,\ldots,18$, on the Square Lattice

N	$N^2 c_N R_\perp^2(N)$
1	0
2	1
3	14
4	126
5	880
6	5216
7	27584
8	134482
9	617918
10	2715810
11	11533208
12	47657874
13	192595952
14	764040260
15	2983906774
16	11498093742
17	43793769160
18	165109451636

We conjecture that this expression is exact without the $o(z^{16})$ contribution. The asymptotic behaviors (1.1)–(1.2) correspond, via (5.2), to the $(1-\lambda z)^{\theta-2\nu_\perp-3}$ singularity in $W(z)$. Thus, we obtain the result

$$\nu_\perp = 1/2 \tag{5.5}$$

Both the exponent values and the form of the finite-size scaling function $P(\tau)$ for our problem are similar to those of the partially directed self-avoiding walk on the square lattice.[5,8–13] Finite-size results for walks have been reported in Refs. 5 and 8. Indeed, the structure of compact lattice animals is in many respects walklike. The value $\nu_{||}=1$ suggests that there is a finite number of sites per typical fixed-X column. The perpendicular fluctuations are Gaussian ($\nu_\perp = 1/2$) and result from the wandering of the column locations along the Y axis, not from the "breathing" of the columns with respect to each other by variation of their widths.

Finally, we note that the result $\nu_\perp = 1/2$ can be confirmed by considering the second moment of the lowest, or the average, Y coordinate of the *last* column of each animal, averaged over all N-site animals. The appropriate generating functions for these quantitites satisfy recursion relations that are too complicated to handle analytically. However, the general form can be analyzed to confirm that the functions

$$\sum_{N=1}^{\infty} [N c_N r_{\perp}^2(N)] z^N \sim (1-\lambda z)^{\theta - 2\nu_{\perp} - 2} \tag{5.6}$$

have singularity $\sim (1-\lambda z)^{-3}$. Here $r_{\perp}^2(N)$ is a "last column" Y-cluster-size measure as defined above. Details of these rather technical considerations are not reported in this paper.

6. TRIANGULAR LATTICE MODEL

It is of interest to have several exactly solvable variants of the directed compact lattice animal model. Indeed, one would like to test the critical exponent universality and compare the global structure of the generating functions. We recall that the fully directed square lattice model[3,1] has not been solved exactly. On the triangular lattice directed according to the rule of Fig. 1, exact solution is possible. We consider here only the number generating functions $F_k(z, 1)$ and $A(z, 1)$ (notation of Section 3). The second argument, 1, will be omitted for brevity.

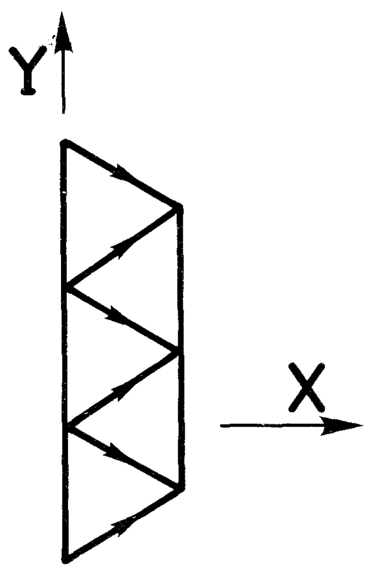

Fig. 1. Triangular lattice directed along two of the three principal axes. Five lattice cells are shown.

Considering the connectivity of the triangular lattice (Fig. 1), one can easily find the analog of relation (2.9) for this lattice,

$$F_k(z) = 1 + \sum_{m=1}^{\infty} (k+m)\, z^m F_m(z) \tag{6.1}$$

One can also check that (2.10) and (2.11) apply. Calculation of the apropriate functions $A(z)$ and $B(z)$ proceeds as in Ref. 1: (2.11) is substituted in the $k=1$ and $k=2$ relations (6.1). We obtain a system of two linear equations,

$$A + B = 1 + \sum_{m=1}^{\infty} (m+1)\, z^m (mA + B) \tag{6.2}$$

$$2A + B = 1 + \sum_{m=1}^{\infty} (m+2)\, z^m (mA + B) \tag{6.3}$$

After evaluating the sums over m, a straightforward algebra yields

$$B(z) = \frac{(1-z)^2(1-3z+z^2)}{1-6z+10z^2-7z^3+z^4} \tag{6.4}$$

$$A(z) = \frac{z(1-z)^3}{1-6z+10z^2-7z^3+z^4} \tag{6.5}$$

The singularity nearest to the origin is a simple pole at $z_c = \lambda^{-1}$, where, numerically,

$$\lambda = 3.86313074324\ldots \tag{6.6}$$

Thus, $\theta = 0$, as expected, and we also note that the structure of the solution is very similar to the square lattice model.[1]

7. HIGH-DIMENSIONAL MODELS

Consider the d-dimensional hypercubic lattice with coordinate axes X, Y_1, Y_2,..., Y_{d-1}. Here X is the directed, "time" axis, while the connectivity along all the Y_j axes is two-way. There are several possible definitions of compactness, all essentially restricting the allowed shape of the cluster cross section at each time level, i.e., for fixed X. We chose the following rule: at each $X > 0$, the cluster sites form a *straight line* (root) of nearest neighbor sites parallel to one of the Y_j axes. To avoid multiple counting of certain translationally and 90°-rotationally equivalent clusters, we require that at

$X=0$ the root is along the Y_1 axis with the site of lowest Y_1 at the origin ($Y_j=0$). The connectivity rule is that each cluster site must be reachable through other cluster sites from the origin, by a walk of $+X$ and $\pm Y_j$ ($j=1, 2,..., d-1$) steps.

As in Section 6, we restrict our consideration to the k-root cluster number generating functions $F_k(z)$. Relation (2.9) is extended to read

$$F_k(z)=1+\sum_{m=1}^{\infty}[k+m-1+km(d-2)]\,z^m F_m(z) \tag{7.1}$$

where the new $km(d-2)$ term accounts for the possibility of 90° rotated roots parallel to $Y_2, Y_3,..., Y_{d-1}$ at $X=1$. Relation (2.11) is valid here, but the identification of the full generating function with $A(z)$, as in (2.10), is no longer correct. The functions $A(z)$ and $B(z)$ can be calculated as in Ref. 1, yielding

$$B(z)=\frac{(1-z)[(1-z)^3-(d-2)\,z(1+z)-z(1-z)]}{1-(d+3)z+7z^2-4z^3} \tag{7.2}$$

$$A(z)=\frac{z(1-z)^2(d-1-z)}{1-(d+3)z+7z^2-4z^3} \tag{7.3}$$

Let us denote the full generating function by $T(z)$; then

$$\begin{aligned} T(z)&=\sum_{N=1}^{\infty}c_N z^N=\sum_{n=1}^{\infty}z^n F_n(z) \\ &=\sum_{n=1}^{\infty}z^n[nA(z)+B(z)] \\ &=\frac{z[(1-z)^3-(d-2)z^2]}{1-(d+3)z+7z^2-4z^3} \end{aligned} \tag{7.4}$$

For "physical" dimensionality values $d=2, 3,...$ the singularity nearest to the origin is a simple pole at λ^{-1}, where $\lambda(d)$ is given by

$$\begin{aligned} \frac{12}{\lambda(d)}=7&+\{[(12d-13)^3+(126d-181)^2]^{1/2}-(126d-181)\}^{1/3} \\ &-\{[(12d-13)^3+(126d-181)^2]^{1/2}+(126d-181)\}^{1/3} \end{aligned} \tag{7.5}$$

Thus, $\theta=0$ for all physical d values. The nondegenerate, positive real root of the denominator of (7.4) exists for all $-\infty<d<+\infty$. Note, however, that for general d the use of (7.5) is not always straightforward; complex branch interpretation is needed. The singularity in the generating function can be "continued" to any d. However, for

$$d<d_0=1.40972... \tag{7.6}$$

other roots of the denominator of (7.4) move closer to the origin of the z plane. [The numerical value in (7.6) is computer-generated.] As a result, the asymptotic form of c_N [see (7.4) and (1.1) with $\theta = 0$],

$$c_N \sim \lambda^N(d) \tag{7.7}$$

is valid only for $d > d_0$. The fixed-cluster-size N ensemble is less suited for the continuation in dimensionality than the fixed-fugacity z grand-canonical ensemble.

The large-d behavior of $\lambda(d)$ can be calculated from (7.5). We find

$$\lambda(d) = d + 3 + O(1/d) \tag{7.8}$$

The proportionality to d, which in turn gives the lattice coordination number [$(2d-1)$ for the one-axis directed hypercubic lattice] is a rather general feature of cluster statistics models.

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